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MATHEMATICS AND LIFE INSURANCE.¹By PERCY C. H. PAPPS,² Newark, N. J.

For an actuary to prepare an address for an association of mathematical teachers is in many respects more difficult than to prepare a paper for a society of actuaries. The thought has occurred to me that it might be interesting if I endeavor to show to what extent mathematics is used in actuarial work. I know that there is a very general idea that an actuary must be a skilled mathematician. In fact, it may not be too much to say that in many instances it is taken for granted that a skilled mathematician must of necessity make an expert actuary. Except for unusual computations, an elementary knowledge of mathematics is all that is required to understand the actuarial formulæ used in daily work. Some of the simpler formulæ and the manner in which they are derived will first be described and I will then outline some of the applications of the higher mathematics to life insurance work.

One of the most common fallacies which we meet is that the expectation of life is used in actuarial calculations. So far as I am aware, the only use which is ever made of the expectation of life is in comparing the effects of mortality under different mortality tables. It is rather generally supposed that in order to arrive at the single premium for an insurance of \$1,000, payable upon the death of a man age 35, we take the expectation of life at age 35 and ascertain what sum of money now invested will at the end of the term of years equal \$1,000 at the assumed rate of interest. A single premium thus computed is smaller than the true single premium. This can be proved mathematically, but it can be readily proved by general reasoning. In the first place the expectation of life of a man age 35 is the mathematical mean of the number of years lived by a large number of men now of age 35. According to the "American Experience Table of Mortality," the expectation of life at age 35 is about thirty-two years, and it may be supposed that in the computations there is one life that lived twenty-two years, and one which lived forty-two years, the mean of these two lives being thirty-two years. At 3% interest \$521.89 is the present value of \$1,000 due at the end of twenty-two years, and \$288.96 is the present value of \$1,000 due at the end of forty-two years. \$810.85 is therefore the amount that would be required to be placed at interest at 3% in order to pay the two claims of \$1,000 each at the end of twenty-two and forty-two years respectively. If in place of using the actual assumed duration for these two lives we used the expectation of life, which is thirty-two years, we would find that \$388.34 is the present value of \$1,000 at the end of thirty-two years, or \$776.68 is the present value of \$2,000 payable at the end of thirty-two years. The present value of these two cases when the expectation of

¹ Read before the Association of Mathematics Teachers of New Jersey, November 20, 1915. Two sections enclosed between asterisks * were added in proof sheets.

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life is used is therefore \$34.17 less than the correct amount. As a matter of fact, the expectation of life could only be correctly used if we assumed interest at the rate of 0%. When we assume any interest we are dealing with a geometrical rather than an arithmetical progression, and as stated before, the expectation of life is an arithmetical mean.

To find the single premium for \$1,000 of insurance on a life now age 35, we require to know the rate of interest which it is safe to assume will be earned by the company for many years to come, and it is also necessary that we should have a table showing the rates of mortality likely to be experienced in the future, or, to put it perhaps more correctly, a table of mortality showing rates which will be not less than those likely to be experienced in the future. Before proceeding further it may be of interest briefly to outline the method employed in compiling mortality tables.

In the first place, by ascertaining for each age of life how many lives, how many policies, or how much insurance, as the case may be, was exposed to the risk of death, and how many lives, policies, or how much insurance was actually cancelled by death, we can ascertain for each age the rate of mortality. Even with a large amount of data, these mortality rates will exhibit accidental fluctuations, so that the data as first ascertained must be suitably graduated. It is not sufficient to ascertain the rate of mortality at each age, for it is necessary that we should have a table to show us out of a certain number living at one age how many will be living at some future age. This is done by taking an arbitrary number, such as 100,000 lives, at age ten, let us say, and by the rate of mortality at age ten we find out the number dying between ages ten and eleven. Subtracting this from 100,000, we have the number living at age eleven. Proceeding in this manner, we have a table showing the number living and the number dying at each age. We are then in a position to proceed to the calculation of single premiums and annuities. It may be mentioned in passing that the rate of mortality is not the only function or the usual function used in the graduation of a mortality table. It would be possible, for example, to start with a fixed radix and use the ungraduated rates to find the number living at each age, and then graduate the figures thus ascertained.

To find the single premium for an insurance of unity on a life now of age 35, we have theoretically to ascertain the cost of providing insurance between ages 35 and 36, between ages 36 and 37, and so on, until we come to the last age shown in the mortality table. In other words, we have a large number of computations to make, the sum of which will give us the single premium required. From our life table we know the number dying between ages 35 and 36. It is assumed that the claims will be paid at the end of the year. If we discount the amount required to the beginning of the year and divide this by the number of lives shown by the mortality table to be living at age 35, we would have the cost of insurance on a single life table between ages 35 and 36. We can ascertain the number dying between ages 36 and 37, discount for two years the amount required to pay these claims, divide by the number living at age 35, and we have the cost of providing

the insurance between ages 36 and 37. This is the way in which the computations are made in theory. As a matter of fact, by means of what are known as commutation tables the calculations are made very expeditiously.

The commutation tables consist of several columns. If we take the number dying between ages 35 and 36 and multiply this by the present value of a unit payable thirty-six years hence, we will find the value of the C column for age 35. If we multiply the number living at age 35 by the present value of a unit due at the end of thirty-five years, we will have the value of the D column for age 35. If we then divide the value of the C column by the value of the D column for age 35, we will have the probability of a life age 35 dying between ages 35 and 36, discounted for one year. We may start from the bottom of the C column and form what is known as the M column by continuous addition of the values in the C column. Opposite age 35 in this column we will then have the sum of the values for the C column from age 35 to the end of the table. If this value in the M column is divided by the value for age 35 in the D column, we will immediately have the single premium for an insurance of unity at age 35. A similar summation of the D column gives what is known as the N column, and dividing any value in the N column by the corresponding value in the D column gives the value of a life annuity for the age selected. If, for example, we wished to obtain the value of an annuity running for ten years on a life age 35, we could take the sum of the values in the D column for ages 35 to 44 inclusive, and divide this by the value in the D column at age 35. It is a simpler matter, however, to subtract from the value in the N column at age 35 the value in the N column at age 45, which will immediately give us the sum of the ten values in the D column.

It has been explained that the single premium for an insurance at age 35 is merely the sum of the several costs of providing insurance payable in the event of a life dying in every year of life, and it is of course impossible that the event can happen more than once. In the same way, the value of an annuity of unity a year is the sum of the present values of the probabilities of a life living to each age to the end of the mortality table.

There are comparatively few who can afford to buy their insurance by means of a single cash payment, and a large proportion of the policies sold are what are known as Ordinary Life policies, where the premiums are payable during the continuance of the contract. Knowing the single premiums and life annuities, it is a simple matter to ascertain the annual premium required. It may be supposed that a company sells \$1,000 of insurance and that the policyholder sells to the company a life annuity of a certain amount. The present value of the one must be equal to the present value of the other. If, therefore, we divide the single premium by the value of a life annuity of unity, the quotient will be the annual premium required. Under a Twenty Payment Life policy the premiums are limited to twenty years, so that the annual premium is ascertained by dividing the single premium by the present value of an annuity of unity running for twenty years. It may be mentioned that under an ordinary annuity the payments to the annuitant are made at the end of each year; under a life insurance

contract the premiums are paid at the beginning of each year. Care must be taken, therefore, that in calculating the value of annual premiums we use an annuity providing for the payments at the beginning of each year. On account of this difference, there are two forms of the N column of the commutation tables, which are likely to confuse the student until the difference is recognized.

The calculation of single and annual premiums for endowment insurances offers no difficulties. It must be remembered that a Twenty-Year Endowment insurance is made up of two parts. We have Twenty-Year Term insurance and what is known as a Twenty-Year Pure Endowment. Under the term insurance the claim is paid if the life dies during the twenty years, otherwise the contract becomes valueless at the end of twenty years. Under the Pure Endowment the contract is valueless in the event of the death of the insured, but is paid in full if the insured survives the twenty years. The single premium for the term insurance is ascertained for age 35 by taking the sum of twenty values in the C column from age 35 to 54 inclusive, and dividing by the value in the D column for age 35. The sum of the values in the C column is of course ascertained by subtracting the value in the M column at age 55 from the value in the same column for age 35. The single premium for the pure endowment at age 35 is ascertained by dividing the value in the D column for age 55 by the value in the same column for age 35.

We now come to the question of reserve. We have in the business of life insurance a very good illustration of the old saying, that a little knowledge is a dangerous thing. If the functions of the reserve on a life insurance policy had been well understood, this country would have been saved the enormous dissatisfaction and hardships caused by the failure of assessment insurance and fraternal orders. These concerns operated on what is known as the "reserve in the pocket" plan. That is, the members were supposed to keep the reserve in their own pockets. This will be referred to later. The reserve on a life insurance policy may be explained in several different ways. It has already been shown that at the inception of a contract of life insurance on the Ordinary Life plan, the single premium is exactly equal to the present value of the annual premiums. Consider the conditions of affairs when the policy has been in force one year. The value of the insurance granted by the company is greater than it was at the inception of the contract, for the insured is one year older, and consequently one year nearer his death. On the other hand, the value of the future premiums which the insured has agreed to pay to the company has decreased for the reason that the insured is one year older and certainly has one less premium to pay. The company's liability represented by the present value of the insurance granted is therefore in excess of the company's asset, which is represented by the present value of the annual premiums yet to be received. Therefore, in order that the company may not be insolvent, it must have on hand a sum equal to the difference between the value of the insurance granted and the value of the premiums yet to be received. This is technically known as the reserve.

The question may arise as to where the reserve comes from. It is readily seen that as a man grows older the yearly cost of providing insurance, provided a

new contract is taken each year, must steadily increase. Under an Ordinary Life policy it does not increase for the reason that the cost is equalized. The first premiums are more than sufficient to pay for carrying the insurance, and it is the balance that goes into the reserve. Furthermore, in the event of death the company is called upon to provide also the difference between the face of the contract and the reserve. As the life grows older, the reserve steadily increases, so that the amount at risk steadily decreases. It thus comes about that in spite of the increasing age of the insured, the annual premium plus the interest on the reserve is always sufficient to pay for the yearly cost of insurance and provide for the necessary increase in the reserve. At age 96, which is the limit of life according to the American Table of Mortality, the entire reserve exactly equals the face of the policy.

The assessment plan of insurance is correct in theory, for there is no theoretical reason why a policyholder should not pay each year just the mere cost of providing his insurance. The objections are practical ones. The net premium at 3 per cent. interest for one year's insurance of \$1,000 at age 35 is \$8.68. At age 75 it is \$91.62. If a man were so fortunate, or unfortunate, as to live ten years more, he would have to pay for one year's insurance \$228.69. Many assessment societies have attempted to provide insurance at level rates at an amount more than sufficient to cover the cost at the young ages, but without holding any reserve which would enable them to continue that cost at the older ages. They have claimed that they left the reserve in their members' pockets. If that was the case it was permissible for them to allow the members to withdraw and retain the reserve in their pockets. The great mistake which they made was that when members died they forgot all about the reserve being in the members' pockets, and paid the full amount of the certificates to the beneficiaries. It is quite possible, if not probable, that many fraternal societies which have been wound up on account of insolvency, would be today in existence if they had paid the beneficiaries merely the difference between the face of the certificates and the amount which the deceased members had retained as reserve in their pockets.

From what has been said you will readily see that a knowledge of elementary algebra up to and including the progressions is sufficient to enable anyone to grasp the ordinary formulæ necessary for calculating premiums and reserves. We will now pass on to consider a few cases where some knowledge of the higher mathematics is necessary. From what has been said it will readily be seen that the calculation of a single premium, for example, where we have merely the mortality table without any commutation columns would be an exceedingly laborious undertaking. Let us suppose that we wish to calculate the cost of an annuity on the life of a woman after the death of her husband. Now in the first place, we know that the mortality on the life of an annuitant is very different from the mortality on the life of one who is insured. It is necessary, therefore, to suppose that our prospective annuitant will experience a mortality according to an annuity table, while her husband will be one whose mortality we will have to measure by a table on which our premium rates for life insurance are

based. We are therefore forced to consider a computation which will involve two mortality tables as well as a rate of interest. In such a case the integral calculus is of inestimable benefit. By means of the calculus we can use a formula of approximate summation which will give results surprisingly close to an exact calculation. The use of one of these formulæ can be best described by taking a simple example of a single premium for life insurance on a single life. It has already been shown how by exact calculation we can ascertain the present value of a life age 35 dying in the first, second, third years, etc., from the present time, and add all these values together. By means of the approximate summation formulæ we can ascertain the present value of the probability of a life dying in the eighth year, the twenty-fourth, fortieth, forty-eighth and fifty-sixth years; multiplying each of these values by proper coefficients and taking eight times the sum of the results will give us a practically correct result. It is true that the approximate summation formulæ are based on the integral calculus, and to understand the derivation of the formulæ a knowledge of the differential and integral calculus is required; yet, as a matter of fact, these approximate summation formulæ are in daily use by those who have no knowledge of either the differential or integral calculus, but who have been trained to apply the formulæ.

* Starting with the well known Maclaurin's Theorem

$$u_x = u_0 + x \frac{du_0}{dx} + \frac{x^2}{2} \frac{d^2u_0}{dx^2} + \frac{x^3}{3} \cdot \frac{d^3u_0}{dx^3} + \text{etc.}$$

and integrating we get

$$\int u_x dx = xu_0 + \frac{x^2}{2} \frac{du_0}{dx} + \frac{x^3}{3} \frac{d^2u_0}{dx^2} + \frac{x^4}{4} \frac{d^3u_0}{dx^3} + \text{etc.}$$

Taking the integral between the limits $-n$ and $+n$

$$\int_{-n}^n u_x dx = 2nu_0 + \frac{2n^3}{3} \frac{d^2u_0}{dx^2} + \text{etc.} \quad (A)$$

All but the first two terms on the right hand side of the above equation may be ignored and it may be assumed that

$$\int_{-n}^n u_x dx = au_0 + b(u_{-n} + u_n). \quad (B)$$

Expanding by Maclaurin's Theorem this becomes

$$\int_{-n}^n u_x dx = au_0 + b \left(2u_0 + \frac{2n^2}{2} \frac{d^2u_0}{dx^2} + \text{etc.} \right). \quad (C)$$

Equating the coefficients of expressions (A) and (C) it will be seen that $a+2b=2n$ and $bn^2 = n^3/3$. From this it follows that $a = 4n/3$ and $b = n/3$. Substituting these values in (B) we get

$$\int_{-n}^n u_x dx = \frac{n}{3} (u_{-n} + 4u_0 + u_n)$$

or

$$\int_0^{2n} u_x dx = \frac{n}{3} (u_0 + 4u_n + u_{2n}).$$

From this it follows that

$$\begin{aligned} \int_0^\infty u_x dx &= \int_0^{2n} u_x dx + \int_{2n}^{4n} u_x dx + \int_{4n}^{6n} u_x dx + \text{etc.} \\ &= \frac{n}{3} \{ (u_0 + 4u_n + u_{2n}) + (u_{2n} + 4u_{3n} + u_{4n}) + \text{etc.} \} \\ &= \frac{n}{3} \{ u_0 + 2(u_{2n} + u_{4n} + u_{6n} + \text{etc.}) + 4(u_n + u_{3n} + u_{5n} + \text{etc.}) \}. \end{aligned}$$

In the example illustrating this formula given in the *Institute of Actuaries' Text Book*, n has been given the value 15 in ascertaining the value of a continuous life annuity at age 30. Instead of computing the probabilities of living every duration from one to sixty-five years only four values are used, namely, $v^{15}l_{45}/l_{30}$, $v^{30}l_{60}/l_{30}$, $v^{45}l_{75}/l_{30}$ and $v^{60}l_{90}/l_{30}$ where l_{30} , for example, represents the number living at age 30 according to the mortality table used in the computations and v^{15} the present value of a unit payable fifteen years hence. The value of the annuity obtained by using the formula is 20.3899 while the true value is 20.3919; an error of only .002.

By taking more terms in formulæ (A) and (B), other summation formulæ may be derived. The one referred to above is thus derived. The formula is

$$\int_0^\infty u_x dx = n(.28u_0 + 1.62u_n + 2.2u_{3n} + 1.62u_{5n} + .56u_{6n} + 1.62u_{7n}).$$

In applying this formula n is so taken that $7n$ falls just within or just beyond the limit of life shown by the mortality table.*

In the graduation of mortality tables many methods are employed. We may take the sum of several values, assume that the average of these values is applicable to the central age of the group, plot the points, and by drawing a smooth curve through the points a series of graduated values may be obtained. The graduation of a set of values may also be accomplished by means of a summation method. For example, if we take three times the sum of three consecutive values from ten times the central value, the result will be a very rough graduation. By taking these values and summing each five adjacent values, repeating this operation three times and taking eight one thousandths of the final result a smoothly graduated series of values will result, unless the ungraduated values are very uneven. An extension of this principle enables us to derive formulæ where the graduated value at any age will be made up of suitable portions of the values extending as far as fourteen terms on each side of the central value. A graduation of a mortality table may also be effected by assuming that a certain mathematical law will hold. One well-known law involves four arbitrary constants, which must

be determined from the ungraduated data. The age, of course, is the variable, and by choosing suitable constants it has been found possible to graduate many mortality tables, so that the number living at any age is determined by the mathematical law. One great advantage which follows the use of one of these mathematical laws is that where several lives are involved in the computations it is possible to substitute for the lives of different ages the same number of lives of equal ages, the corresponding age being ascertained from the graduation constants. In the selection of the graduation constants so that the graduated values may fit as closely as may be to the ungraduated data, application has been made of the theory of moments, the method of least squares and frequency curves.

* If u_0 be a central value and u_{-1} and u_{+1} be adjoining values then $10u_0 - 3(u_{-1} + u_0 + u_{+1})$ is the simple formula mentioned above, which reduces to $\hat{u}_0 = -3u_{-1} + 7u_0 - 3u_{+1}$, where $\hat{u}_0$ is a roughly graduated value of u_0 . A further development of this principle gives Woolhouse's formula, which is as follows:

$$\hat{u}_0 = (1/5)\{u_0 + .96(u_{-1} + u_{+1}) + .84(u_{-2} + u_{+2}) + .28(u_{-3} + u_{+3}) \\ + .12(u_{-4} + u_{+4}) - .08(u_{-6} + u_{+6}) - .12(u_{-7} + u_{+7})\}.$$

The lengthy formula referred to above was described by the author in a paper published in the December 1918 number of the *Publications of the American Statistical Association*. This formula is as follows:

$$\hat{u}_0 = (1/5)\{u_0 + .9344(u_{-1} + u_{+1}) + .7264(u_{-2} + u_{+2}) + .4384(u_{-3} + u_{+3}) \\ + .1744(u_{-4} + u_{+4}) - .0976(u_{-6} + u_{+6}) - .1136(u_{-7} + u_{+7}) \\ - .0736(u_{-8} + u_{+8}) - .0256(u_{-9} + u_{+9}) + .0128(u_{-11} + u_{+11}) \\ + .0144(u_{-12} + u_{+12}) + .0080(u_{-13} + u_{+13}) + .0016(u_{-14} + u_{+14})\}.$$

It will be found that the sum of the coefficients within the bracket is equal to 5.

The simplest form of the mathematical law of mortality is that of Gompertz which, expressed logarithmically, is $\log l_x = \log k + c^x \log g$ where l_x represents the number living at age x and k , c and g are constants.

A first modification of this law derived by Makeham is

$$\log l_x = \log k + x \log s + c^x \log g.$$

A second modification is

$$\log l_x = \log k + x \log s + x^2 \log \omega + c^x \log g.*$$

It will be recognized that I have touched but briefly on many subjects which can only properly be covered in an extensive actuarial library. It should not be forgotten, however, that very few actuaries are ever called upon to graduate a mortality table; consequently it is of more importance to an insurance company that the actuary should be a man of good business judgment, with some original-

ity, able to meet the public, and compose a decent letter. He must have the mathematical ability to enable him to pass his actuarial examinations and qualify as an actuary, but so far as his practical work is concerned, in the training of his actuarial department for the regular routine work of the office he can forget if he will practically all of his higher mathematics. When an actuary is called upon, as is frequently the case, to take his part in the executive work of the company, it is necessary that he should have some knowledge of life insurance law, the principles of accounting, and a general knowledge of investments and finance. From what I have said I believe that you will agree with me when I say that an actuary is not necessarily an expert mathematician, and that an expert mathematician will not necessarily make a good actuary.

NOTE ON THE ROOTS OF THE DERIVATIVE OF A POLYNOMIAL.

By W. H. ECHOLS, University of Virginia.

(Read before the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America, May 15, 1920.)

The problem of determining regions to which are confined the roots of the derivatives of functions has received a great deal of attention. Lucas established the fact¹, by a mechanical proof, that the roots of the derivative of any polynomial are confined to the smallest convex region enclosing the roots of the polynomial. Maxime Bôcher gave a geometrical demonstration of this.² He noted that the roots of the derivative of any cubic equation are the foci of the ellipse tangent at their mid-points to the sides of the triangle of the roots of the cubic; and remarked that it might be possible to associate the roots of the derivative of any polynomial with the foci of a higher plane curve. B. Z. Linfield gave a beautiful demonstration, in a paper read at the meeting, in St. Louis last December, of the American Mathematical Society (*Bulletin*, p. 264), that the roots of the derivative of any polynomial of degree n were the foci of a curve of class $n - 1$ touching at their mid-points the segments joining, two and two, the roots of the polynomial. Recently J. L. W. V. Jensen stated,³ without demonstration,

¹ F. Lucas, "Géométrie des polynômes," *Journal de l'Ecole Polytechnique*, 1879, cahier 46, tome 28, pp. 1-33.—EDITOR.

² M. Bôcher, "Some propositions concerning the geometric representation of imaginaries," *Annals of Mathematics*, March, 1893, vol. 7, pp. 70-72.—EDITOR.

³ In a letter to the writer Dr. J. L. Walsh, of Harvard, says: "Jensen originally stated his theorem without proof, in *Acta Mathematica*, vol. 36 (1912), p. 190. Apparently the theorem remained unnoticed until Professor D. R. Curtiss called attention to it in a paper presented at a meeting of the American Mathematical Society; his abstract was published in the current volume of the *Bulletin*, pp. 61-62. Professor Curtiss presented further results in April (see the June *Bulletin*, p. 392). I also have some further results which I presented to the Society in December; the abstract of which is in the March *Bulletin*, p. 259. My paper has been accepted for publication in the *Annals of Mathematics*; it contains a proof of Jensen's theorem based on mechanical considerations. Your own proof is surely different from mine (in form but not in substance). I do not know whether it is different from Curtiss's. I was not aware an algebraic proof could be given so simply. No proof of Jensen's theorem has yet been published."